WEAKLY NONLINEAR ANALYSIS OF THE EFFECT OF SHELL STIFFNESS ON THE FUNDAMENTAL RESONANCE OF COATED MICROBUBBLES

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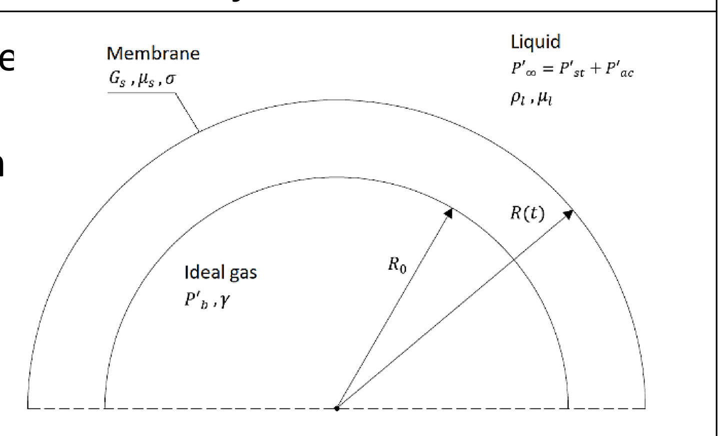
Motivation

- Investigate the weakly nonlinear radial oscillations of a gas bubble encapsulated in an elastic shell. The bubble is immersed in an infinite, slightly compressible Newtonian fluid and is subject to a sinusoidal acoustic excitation in the far field.
- The viscoelastic properties of the shell are described by the appropriate constitutive law:
  ➢ Kelvin-Voight for an almost linear material ➢ Mooney-Rivlin for a strain softening material ➢ Skalak for a strain hardening material
- In an attempt to reconcile discrepancies between available simulations[1] and asymptotic analysis[2],[3]:
  - ➢ A different approach to the scaling of the nonlinear terms is employed
  - ➢ Steady-state solutions corresponding to the fundamental resonance of the bubble are sought, valid to second order of approximation in terms of the sound amplitude
  - ➢ The asymptotic scheme is validated against previous numerical studies[1].

Schematic representation of the microbubble

- Acoustic disturbance $P'_{ac} = \eta P'_{st} sin(\omega_f t')$
- Adiabatic expansion $P_{b,0}V_0^{\gamma} = P_b V^{\gamma}$
- Young – Laplace pressure coupling $P_{b,0} = P_{st} + \frac{2\sigma}{R_0}$



Mathematical formulation and asymptotic analysis

- Keller – Miksis model describing the nonlinear oscillation of the bubble interface[1]

$$(1 - M\dot{R})\ddot{R}R + \left(\frac{3}{2} - \frac{M\dot{R}}{2}\right)\dot{R}^2 = (1 + M\dot{R})(P_l - P_{st} - P_{ac}) + MR\frac{d}{dt}(P_l - P_{ac})$$

- Liquid pressure on the interface for each constitutive law [1]

$$\text{KV} \quad P_l|_{r=R} = \left[\frac{1}{R^{3\gamma}} - \frac{2}{We}\frac{1}{R} - \frac{4}{Re}\frac{\dot{R}}{R} - \frac{4m}{Re}\frac{\dot{R}}{R^2} - 2\frac{3G}{R}(R^2 - 1)\right]$$

$$\text{MR} \quad P_l|_{r=R} = \left[\frac{1}{R^{3\gamma}} - \frac{2}{We}\frac{1}{R} - \frac{4}{Re}\frac{\dot{R}}{R} - \frac{4m}{Re}\frac{\dot{R}}{R^2} - 2\frac{G}{R}\left(1 - \frac{1}{R^6}\right)\right]$$

$$\text{SK} \quad P_l|_{r=R} = \left[\frac{1}{R^{3\gamma}} - \frac{2}{We}\frac{1}{R} - \frac{4}{Re}\frac{\dot{R}}{R} - \frac{4m}{Re}\frac{\dot{R}}{R^2} - 2\frac{G}{R}(R^6 - 1)\right]$$

- Dimensionless numbers

$$Re_l = \frac{\rho R_0 \sqrt{P_{b,0}/\rho}}{\mu_l} \qquad We = \frac{R_0 P_{b,0}}{\sigma}$$

$$m = \frac{3\mu_s \delta}{\mu_l R_0} \qquad G = \frac{\delta G_s}{R_0 P_{b,0}}$$

$$M = \sqrt{\frac{P_{b,0}}{\rho}}\frac{1}{c}$$

- Oscillating bubble radius

$$R(\tau) = 1 + R_d(\tau)$$

- Small nonlinear terms assumption, near main resonance case and scaling

$$\eta P_{st} = \xi = \varepsilon^2 P \ , \ R_d = \varepsilon R_D \ , \ M = \varepsilon\mu \ , \ ^1/_{Re} = \varepsilon r \ , \left(\frac{\omega_0}{\omega}\right)^2 = 1 + \varepsilon\omega_1$$

- Keller – Miksis model with parameter ordering for the near main resonance case

$$\ddot{R}_D + R_D = -\frac{\varepsilon P}{\omega_0^2}sin(T) + \varepsilon^2\left[-\frac{\mu P}{\omega_0}cos(T) - \frac{P}{\omega_0^2}\omega_1 sin(T) + \frac{PR_d}{\omega_0^2}sin(T)\right] + \varepsilon\left\{-\dot{R}_D\left[\mu\omega_0 + \frac{4r(1+m)}{\omega_0}\right] + \frac{\alpha_1 R_D^2}{\omega_0^2} - \frac{3}{2}\dot{R}_D^2 - \omega_1 R_D\right]$$
$$+ \varepsilon^2\left\{-\dot{R}_D\left[\frac{\mu\omega_0\omega_1}{2} - \frac{2r\omega_1}{\omega_0}(1+m)\right] - 4r\mu\ddot{R}_D(1+m) + \frac{\dot{R}_D R_D}{\omega_0}[(9\gamma^2 - 3\gamma)\mu - 12G\mu + 4r(2+3m)] - \frac{\alpha_2}{\omega_0^2}R_D^3 + \frac{3}{2}\dot{R}_D^2 R_D + \frac{\alpha_1 R_D^2}{\omega_0^2}\omega_1\right\} + O(\varepsilon^4)$$

- Power series expansion $R_D(\varepsilon, T) = R_{D_0}(T) + \varepsilon R_{D_1}(T) + \varepsilon^2 R_{D_2}(T) + \cdots$
- Periodicity condition $R_{D_i}(T + 2\pi) = R_{D_i}(T)$, $i = 0,1,2,\ldots$
- Amplitude of the radius deviation

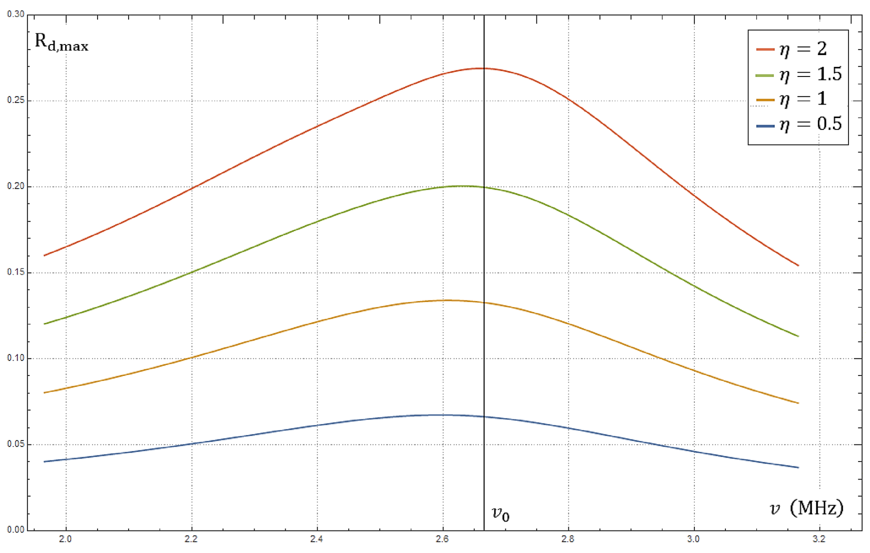
$$\frac{R_{max} - R_0}{R_0} = R_{d,max} = \varepsilon R_{D,max} = \varepsilon\sqrt{(A_0 + \varepsilon A_1)^2 + (B_0 + \varepsilon B_1)^2}$$

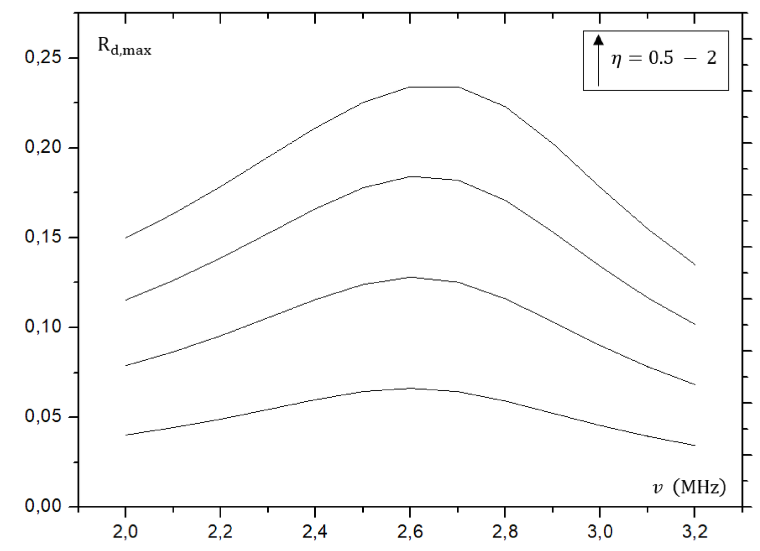
Coefficients A_i B_i calculated by applying the periodicity constraint and negating unbounded terms arising from the resonance condition

Results

Calculations have been conducted for water as the ambient liquid at 20°C with parameters: $\rho_l = 998\, ^{kg}/_{m^3}$, $\mu_l = 0.001\, ^{kg}/_{ms}$ $c_l = 1500\, ^{m}/_{s}$, $\sigma = 0.072\, ^{kg}/_{s^2}$, $\delta \approx 15nm$, $R_0 = 3\mu m, G_s = 35MPa, \mu_s = 0.6\,\frac{kg}{ms}$

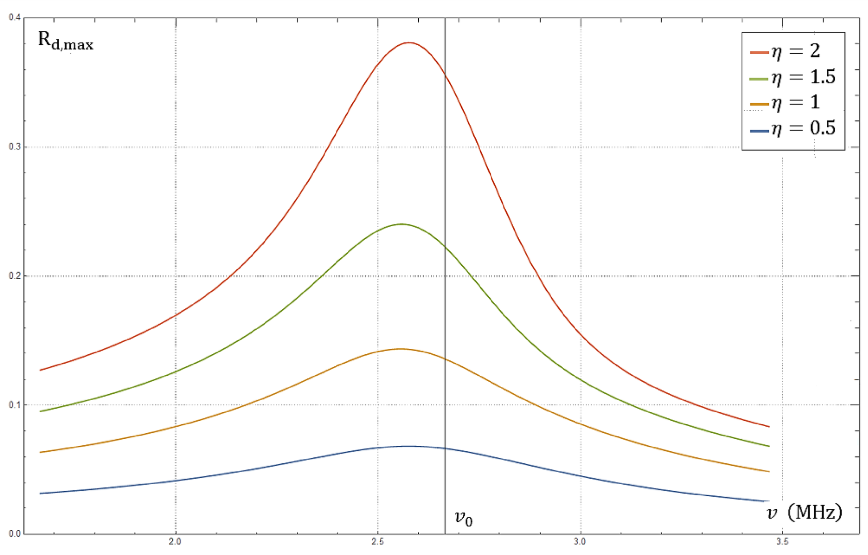
- *Kelvin – Voight membrane*
  - ➢ Linear stress strain behavior valid for small displacements.
  - ➢ Mild effect of the shell resistance on the response amplitude.
  - ➢ Resonance frequency shift towards lower values.
  - ➢ Possible resonance in lower forcing frequencies with proper adjustment of the disturbance amplitude.

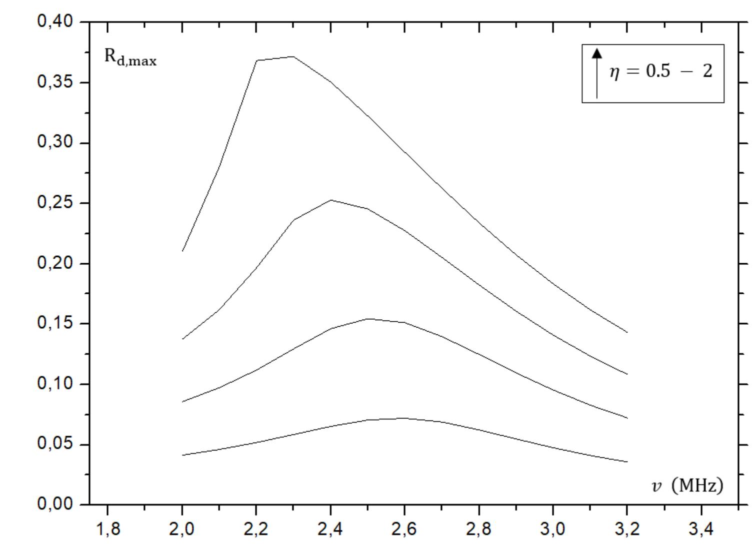




	Analytical		Numerical	
η	$R_{d,max}$	V_{res}(MHz)	$R_{d,max}$	V_{res}(MHz)
0.5	0.06727	2.59	0.06634	2.6
1	0.13399	2.61	0.1281	2.6
1.5	0.20048	2.63	0.1843	2.6
2	0.26891	2.66	0.2314	2.7

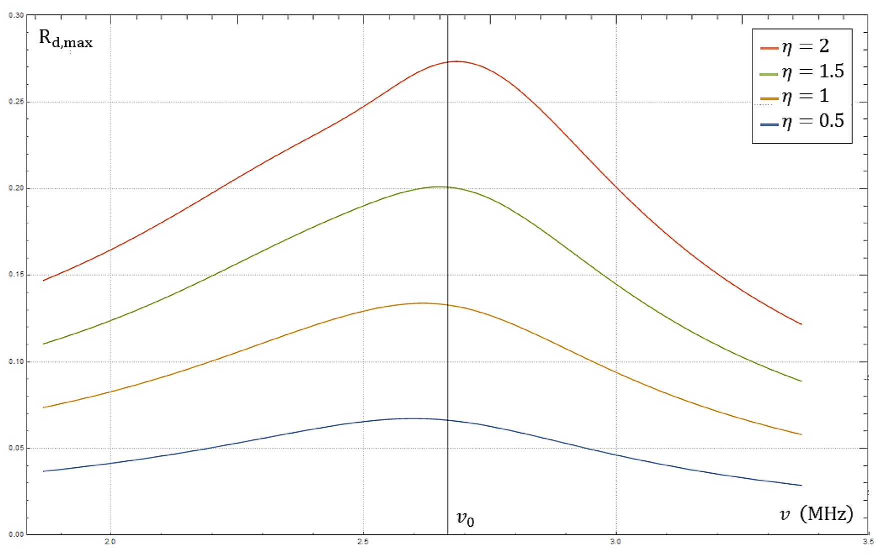
- *Mooney – Rivlin membrane*
  - ➢ Strain-softening nonlinear material.
  - ➢ Enhanced amplitude response due to the progressive softening of the shell.
  - ➢ Resonant responses shift to lower frequencies.
  - ➢ Possible resonance in lower forcing frequencies with proper adjustment of the disturbance amplitude.

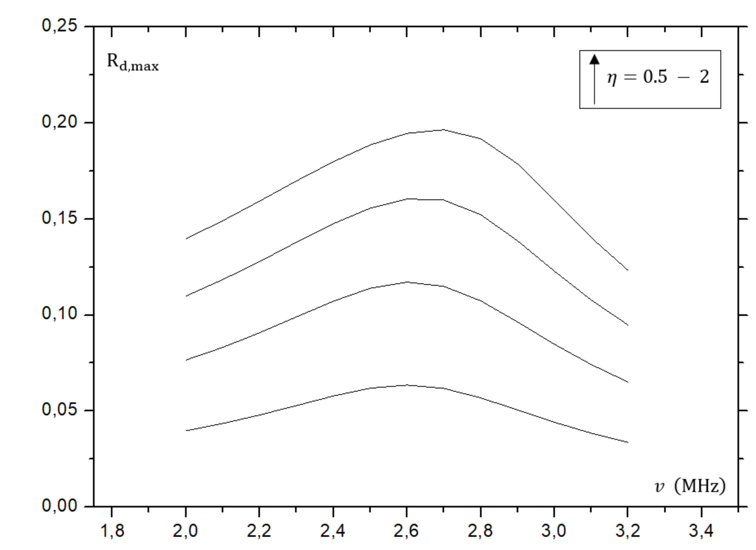




	Analytical		Numerical	
η	$R_{d,max}$	V_{res}(MHz)	$R_{d,max}$	V_{res}(MHz)
0.5	0.06819	2.58	0.07212	2.6
1	0.14354	2.55	0.15464	2.5
1.5	0.24013	2.56	0.25921	2.4
2	0.38081	2.57	0.37183	2.3

- *Skalak membrane*
  - ➢ Strain-hardening nonlinear material.
  - ➢ Diminished amplitude response due to the progressive hardening of the shell.
  - ➢ Resonant responses shift to higher frequencies.
  - ➢ Possible resonance in higher forcing frequencies with proper adjustment of the disturbance amplitude.





	Analytical		Numerical	
η	$R_{d,max}$	V_{res}(MHz)	$R_{d,max}$	V_{res}(MHz)
0.5	0.06722	2.6	0.0636	2.6
1	0.13378	2.62	0.11715	2.6
1.5	0.20093	2.65	0.1605	2.6
2	0.27335	2.68	0.19652	2.7

Figures: *Frequency response curves for increasing amplitude of the external acoustic excitation: (a)analytical results (b)numerical results (c)numerical tables*

Conclusions

General remarks

- Resonant response depends on the constitutive law used to describe the membrane's material.
- Each case leads to resonance in two ways:
  - ➢ For a given ultrasound amplitude, the driving frequency is adjusted to the proper value to achieve resonance.
  - ➢ For a given ultrasound frequency the amplitude is adjusted accordingly so that the driving frequency matches the shifted resonant frequency.
- In the limit of asymptotically small deformations all cases reduce to an almost linearized behavior as expected.[1],[4]

On the asymptotic analysis

- The nature of the shell affecting the magnitude of the response, as well as the shift of the non-linear resonant frequency is captured by the asymptotic analysis.
- For lower values of the sound amplitude:
  - ➢ Analytical and numerical solutions yield comparable results regarding the amplitude of the oscillation and the value of the resonant frequency.
- For higher values of the sound amplitude:
  - ➢ The two methods start to diverge and the asymptotic analysis slightly overestimates values regarding the amplitude and places the resonant frequency in a smaller interval around the eigenfrequency of the medium.
  - ➢ Graphs produced by the asymptotic method fail to portray the steeper slopes which characterize the non-linear oscillations.
- The validity of the employed method lies on small non-linear terms. When the excitation increases beyond a certain point the scaling of the non-linear terms is not accurate and the method breaks down

Further Research

Future research in the framework of PhD studies will cover the case of a coated microbubble attached to a wall due to intermolecular forces:

- Asymptotic analysis of the static configuration and derivation of a boundary condition on the edge of the free region incorporating the effects of the contact region and excluding it from calculations. Stability analysis and calculation of the resonance frequencies.
- Investigation of the dynamic response to an external acoustic disturbance utilizing a boundary element method on the simplified problem as stated above. Comparison against studies incorporating the contact region and the intermolecular forces acting upon it.
- Finite element method implementation on the ambient liquid in the vicinity of the bubble in conjunction with lubrication theory for the thin water film underneath the contact region, in order to capture microstreaming effects.

References

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